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Latent symmetry and its group theoretical determination

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Latent symmetry of a component of a composite object is defined as that symmetry of a subunit of the component which is not a symmetry of the component but which is a symmetry of the composite. This is illustrated by a geometrical example. A group theoretical procedure is given to determine latent symmetry and is applied to examples.

1. Introduction

Consider the following two-component composite object. The first component is shown in Fig. 1(*a*): it is a two-dimensional isosceles triangle with an arbitrary apex angle $\theta < 90°$. In the coordinate system shown, the symmetry of this triangle is the point group $\mathbf{m_x}$. The second component is shown in Fig. 1(*b*), a second two-dimensional isosceles triangle obtained by reflecting the first component across the *x* axis, *i.e.* by applying the operation m_y to the first component. Fig. 1(*c*) shows the composite object. The symmetry group of this rhombus-shaped composite is the point group $\mathbf{2_z m_x m_y}$.

Consider now the same two-component composite object in the special case where the apex angle $\theta = 90°$. The components are shown in Figs. 2(*a*) and 2(*b*), and the resulting square composite object in Fig. 2(*c*). The point group of this second composite is $\mathbf{4_z m_x m_{xy}}$.

The $\mathbf{2_z m_x m_y}$ symmetry of both composite objects can be understood and predicted by considering the symmetry of the components and the operation used in the construction of the composites: in both cases, m_x is a common symmetry of the components, m_y permutes the components, and 2_z is the product of these two symmetries. The additional symmetry of the square second composite cannot, however, be understood or predicted from the common symmetry of the components and the symmetries which permute the components. In this paper, we shall show that a *partial symmetry* (Weinstein, 1996; Lawson, 1998) of a component can be that additional symmetry of the composite. In particular, we consider symmetry-related subunits of a component, *i.e.* subunits related by the symmetry of the component. We find that such subunits can contain symmetries which are not symmetries of the component but which are symmetries of the composite. It is this type of partial symmetry of a component that has been termed *latent symmetry* by Wadhawan (2000).

In §2, we provide a mathematical model of a composite object and introduce a group theoretical procedure to determine which partial symmetries, *i.e.* symmetries of symmetry-related subunits of components, are latent symmetries. This procedure is then applied to two examples. A third example is given to exemplify the limitations of this procedure in determining the symmetry of the composite. In §3, the concept of symmetrizers in the Curie–Shubnikov principle of superposition of symmetries is related to that of latent symmetry. We discuss possible physical applications of latent symmetries in §4.

2. Latent symmetry

We consider a finite object *A* with intrinsic symmetry $\mathbf{H}$. This finite object *A* is the basic component of a composite *S*, an unordered set of objects constructed by applying a set $\{g_1, g_2, \ldots, g_n\}$, where $g_1 = 1$, of isometries to *A*,

$$S = \{g_1 A, g_2 A, \ldots, g_n A\}. \quad (1)$$

We shall consider three contributions to the symmetry of the composite *S*.

(i) The subset $\{g_1, \ldots, g_s\}$ of isometries of the set $\{g_1, g_2, \ldots, g_n\}$ which leave the composite *S* invariant,

$$gS = \{gg_1 A, gg_2 A, \ldots, gg_n A\} = \{g_1 A, g_2 A, \ldots, g_n A\} = S. \quad (2)$$

This subset $\{g_1, \ldots, g_s\}$ of isometries are those operations used in the construction of the composite from the component *A* which permute the components of the composite. If the set of isometries $\{g_1, g_2, \ldots, g_n\}$ used to construct the composite constitutes a group $\mathbf{G}$, then all elements of $\mathbf{G}$ leave the composite invariant.

(ii) The subgroup $\mathbf{H_s}$ of all elements of $\mathbf{H}$ which leave the composite invariant: in Appendix *A* we define an element *h* of $\mathbf{H}$ as belonging to *the normalizer of* $\{g_1, g_2, \ldots, g_n\}$ *modulo* $\boldsymbol{H}$ if

$$h\{g_1, g_2, \ldots, g_n\}h^{-1} = \{g_1 h_1, g_2 h_2, \ldots, g_n h_n\}, \quad (3)$$

where h_i, $i = 1, 2, \ldots, n$, are not necessarily distinct elements of **H**. It is shown in Appendix B that all elements of **H** belonging to the normalizer of $\{g_1, g_2, \ldots, g_n\}$ modulo **H** constitute the subgroup $\mathbf{H_s}$ of all elements **H** which leaves the composite $S = \{g_1A, g_2A, \ldots, g_nA\}$ invariant. An invariance group of the composite S, owing to the above two types of symmetries, will be denoted by **Inv**(1) where

$$\mathbf{Inv}(1) = \big\langle \{g_1, \ldots, g_s\}, \mathbf{H_s} \big\rangle \tag{4}$$

is the group generated by the set of isometries used in the construction of the composite which permute the components of the composite and the subgroup $\mathbf{H_s}$ of all elements of **H** which leave the composite invariant. For both the composite examples in Figs. 1 and 2, $\{g_1, \ldots, g_s\} = \{1, m_y\} = \mathbf{m_y}$, $\mathbf{H_s} = \mathbf{H} = \{1, m_x\} = \mathbf{m_x}$, and $\mathbf{Inv}(1) = \langle \mathbf{m_y}, \mathbf{m_x} \rangle = \mathbf{2_z m_x m_y}$.

(iii) Latent symmetry, the symmetries of symmetry-related subunits of the basic component which leave the composite invariant: since the symmetry group of the basic component A is **H**, there exists a subunit B and a set of symmetry-related subunits of A such that

$$A = \{h_1B, h_2B, \ldots, h_mB\}, \tag{5}$$

where h_i, $i = 1, 2, \ldots, m$, are the elements of the group **H**, and $h_1 = 1$, the identity element of **H**. We denote by **K** the symmetry group of the subunit B. It is shown in Appendix C that the composite

$$S = \{g_1A, g_2A, \ldots, g_nA\} = \{\ldots, g_ih_jB, \ldots\}, \tag{6}$$

where $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$, is invariant under the subgroup $\mathbf{K_s}$ of **K** of elements which are in the normalizer of the set $\{\ldots, g_ih_j, \ldots\}$, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, modulo **K**. An invariance group of the composite S is then the group **Inv**(2),

$$\mathbf{Inv}(2) = \big\langle \mathbf{Inv}(1), \mathbf{K_s} \big\rangle, \tag{7}$$

the group generated by the elements of the groups **Inv**(1) and $\mathbf{K_s}$.

For both the composite examples in Figs. 1 and 2, one can write $A = \{B, m_xB\}$ and $S = \{B, 2_zB, m_xB, m_yB\}$, where the respective subunits B can be seen in Figs. 1(d) and 2(d). For the composite in Fig. 1, with the arbitrary apex angle, the subunit B has a symmetry group **K** consisting only of the identity element. Consequently, $\mathbf{Inv}(2) = \mathbf{Inv}(1) = \mathbf{2_z m_x m_y}$. For the composite in Fig. 2, with a 90° apex angle, the symmetry group of the subunit B is the group $\mathbf{K} = \mathbf{m_{xy}}$. $\mathbf{K_s} = \mathbf{K}$ since the operation m_{xy} is in the normalizer of $\{1, 2_z, m_x, m_y\}$ modulo **K**. Consequently, we have derived as an invariance group of this composite the group **Inv**(2),

$$\mathbf{Inv}(2) = \big\langle \mathbf{2_z m_x m_y}, \mathbf{m_{xy}} \big\rangle = \mathbf{4_z m_x m_{xy}}. \tag{8}$$

We have shown that the additional symmetries of the composite in Fig. 2, additional to the symmetries of the point group $\mathbf{2_z m_x m_y}$, are related to the symmetry m_{xy} of a subunit of the basic component which is a symmetry of the composite. It is this additional (latent) symmetry of a subunit of the basic component which generates this additional symmetry of the composite.

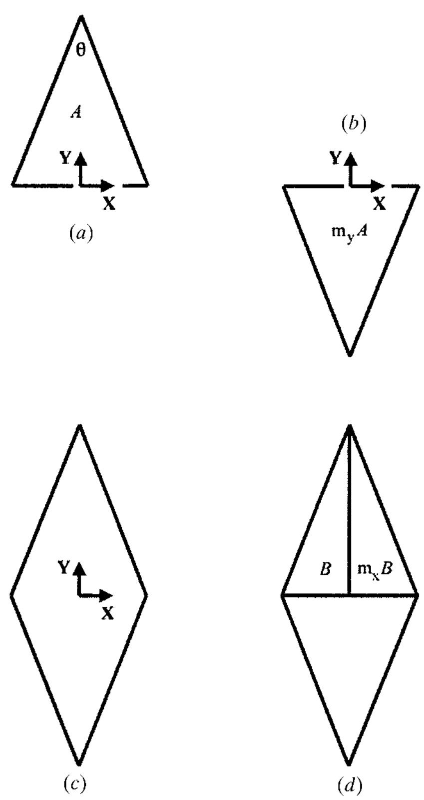

Figure 1
Construction of a rhombus-shaped composite. A planar isosceles triangular basic component A with an apex angle $\theta < 90°$ is shown in (a). A second component m_yA is shown in (b) and the rhombohedral composite in (c). The subunit B of the basic component A is shown in (d).

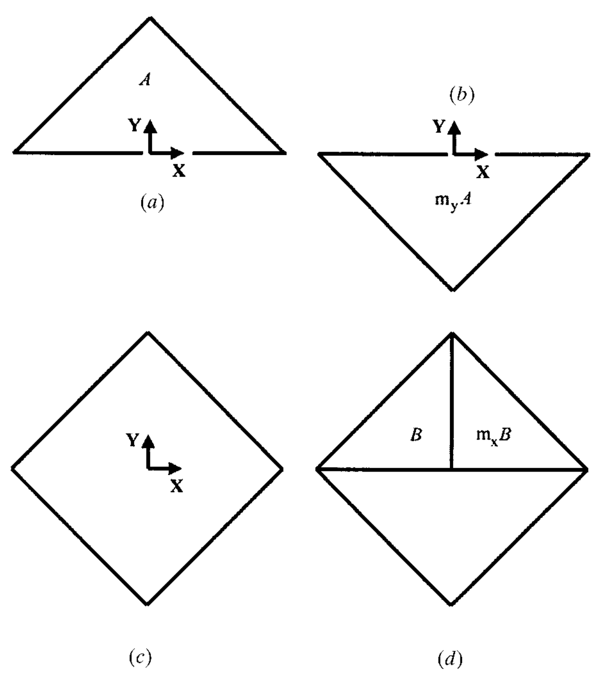

Figure 2
Construction of a square composite. A planar isosceles triangular basic component A with an apex angle $\theta = 90°$ is shown in (a). A second component m_yA is shown in (b) and the square composite in (c). The subunit B of the basic component A is shown in (d).

For the example composite in Fig. 1, the subunit B has only the trivial identity symmetry in its symmetry group. The symmetry group $\mathbf{K}$ of the subunit B does not then imply that the subunit B can be subdivided into smaller subunits. Consequently, in this case, the process of searching for additional latent symmetry in the subunits of the basic component is finished. This is not the case, however, in the square composite in Fig. 2.

If the symmetry group $\mathbf{K}$ of the subunit B is of order greater than unity, there exists a smaller subunit C and a set of symmetry-related subunits of B such that

$$B = \{k_1C, k_2C, \ldots, k_pC\}, \tag{9}$$

where k_q, $q = 1, 2, \ldots, p$, are the elements of the group $\mathbf{K}$ and $k_1 = 1$, the identity element of $\mathbf{K}$. We denote by $\mathbf{L}$ the symmetry group of the subunit C. The composite S,

$$S = \{\ldots, g_ih_jB, \ldots\}, \tag{10}$$

where $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$, can now be written as

$$S = \{\ldots, g_ih_jk_qC, \ldots\}, \tag{11}$$

where $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$; $q = 1, 2, \ldots, p$, and the composite S is invariant under the subgroup $\mathbf{L_s}$ of $\mathbf{L}$ of all elements which are in the normalizer of the set $\{\ldots, g_ih_jk_q, \ldots\}$; $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$; $q = 1, 2, \ldots, p$, modulo $\mathbf{L}$. An invariance group of the composite S is then the group $\mathbf{Inv}(3)$,

$$\mathbf{Inv}(3) = \langle \mathbf{Inv}(2), \mathbf{L_s} \rangle, \tag{12}$$

the group generated by the elements of the groups $\mathbf{Inv}(2)$ and $\mathbf{L_s}$.

If the subunit C has a symmetry group $\mathbf{L}$ consisting only of the identity element, then $\mathbf{Inv}(3) = \mathbf{Inv}(2)$ and the process of search for additional latent symmetry is finished. If the symmetry group $\mathbf{L}$ of the subunit C is of order greater than unity, then the process of defining a smaller subunit and a new invariance group $\mathbf{Inv}$ is repeated. This process continues until the symmetry group of the subunit consists only of the identity element.

We illustrate this continuing process by again considering the composite in Fig. 2. The symmetry group of the subunit B shown in Fig. 2(d) is $\mathbf{m_{xy}}$. Consequently, one can define a subunit C such that $B = \{C, m_{xy}C\}$, see Fig. 3(a). The symmetry group $\mathbf{L}$ of the subunit C consists of the identity and a mirror line perpendicular to the x axis and passing through the apex of the isosceles triangular subunit C. This mirror symmetry is not contained in the normalizer of the set of isometries which generate the composite from the subunit C modulo $\mathbf{L}$. We have then not found any new symmetries of the composite.

Since the subunit C has a symmetry group of order greater than unity, we can repeat the above process, *i.e.* C can be subdivided into smaller subunits. One of these smaller subunits, denoted by D, is shown in Fig. 3(b). Its symmetry group $\mathbf{N}$ consists of the identity and a mirror line perpendicular to the xy direction and passing through the apex of the isosceles triangular subunit D. This mirror symmetry is not contained in the normalizer of the set of isometries which generate the composite from the subunit D modulo $\mathbf{N}$. Again, we have not found any new symmetries of the composite.

The subunit D has a symmetry group of order greater than unity and we could repeat the process and subdivided D. In principle, as in this case, this process is repeated *ad infinitum*, as at no repetition of the process of finding smaller subunits do we find a subunit whose symmetry group consists only of the identity element. For finite composites, however, as in this case, the symmetry group of the composite is a point group and there exists a singular point, line or plane left invariant by this point group. For finite composites, we can apply the following ansatz: *only those isometries which leave invariant the singular point, line or plane of the composite object can be symmetry elements of that composite.* For the finite composite of Fig. 2, we have already found that the composite is invariant under the point group $\mathbf{4_zm_xm_{xy}}$ and a singular point, the origin of the coordinate system, is defined. As the mirror line symmetry of subunits C and D do not pass through this singular point, they cannot be symmetries of the composite. In fact, in this case, no further subdivision into smaller subunits will give rise to a subunit whose symmetry group contains an isometry, except for the identity, which leaves the singular point of the composite invariant. Consequently, a continuation of this process will not reveal any new symmetries of the composite.

As a second example, consider the composite shown in Fig. 4(a). The basic component A is a right triangle with one leg of length 1 and the second of length $\alpha < 1$. The point group $\mathbf{H}$ of the component A consists only of the identity, $\mathbf{H} = \{(1|0,0)\}$, where we have given the identity in two-dimensional Seitz notation, which we use in the symmetry analysis below. The composite $S = \{g_1A, g_2A, \ldots, g_4A\}$ is generated by the set of isometries $\{g_1, g_2, \ldots, g_4\} = \{(1|0,0), (m_y|0,0), (m_{xy}|0,0), (4_z|0,0)\}$. The subset of elements of $\{g_1, g_2, \ldots, g_4\}$ which leave the composite invariant is the subset $\{(1|0,0), (m_y|0,0)\}$, and since the group $\mathbf{H}$ consists only of the identity element, $\mathbf{H_s} = \mathbf{H}$, and, from equation (4), $\mathbf{Inv}(1) = \{(1|0,0), (m_y|0,0)\} = \mathbf{m_y}$. Since the point group $\mathbf{H}$ of the component A is the identity point group, one cannot subdivide the component A into symmetry-related subunits, and consequently the above search for latent symmetry is concluded.

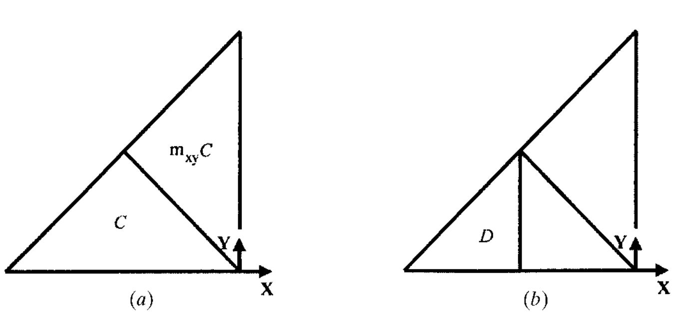

Figure 3
An enlarged view of the subunit B of the basic component A of the square composite, see Fig. 2(d), showing it subdivided into subunits in (a). The subunit C of (a) is further subdivided into smaller subunits in (b).

In Fig. 4(b), we show this composite when $\alpha = 1$, and the component A is now an isosceles triangle. The point group $\mathbf{H}$ of the component A is $\{(1|0,0), (m_{\bar{x}y}|1,-1)\}$, where the translations associated with the mirror plane are due to the fact that the mirror plane does not pass through the origin of the given coordinate system. As the composite in Fig.4(a), the first invariance group of this composite, equation (4), is again $\mathbf{Inv}(1) = \{(1|0,0), (m_y|0,0)\}$.

Since the symmetry group of the component A consists of more than one element, there exists a subunit B and a symmetry-related subunit of A, see Fig. 4(c), such that, see equation (5),

$$A = \{(1|0,0)B, (m_{\bar{x}y}|1,-1)B\}. \qquad (13)$$

The symmetry group of the subunit B is $\mathbf{K} = \{(1|0,0), (m_x|1,0)\}$. The composite S can then be written in terms of this subunit B,

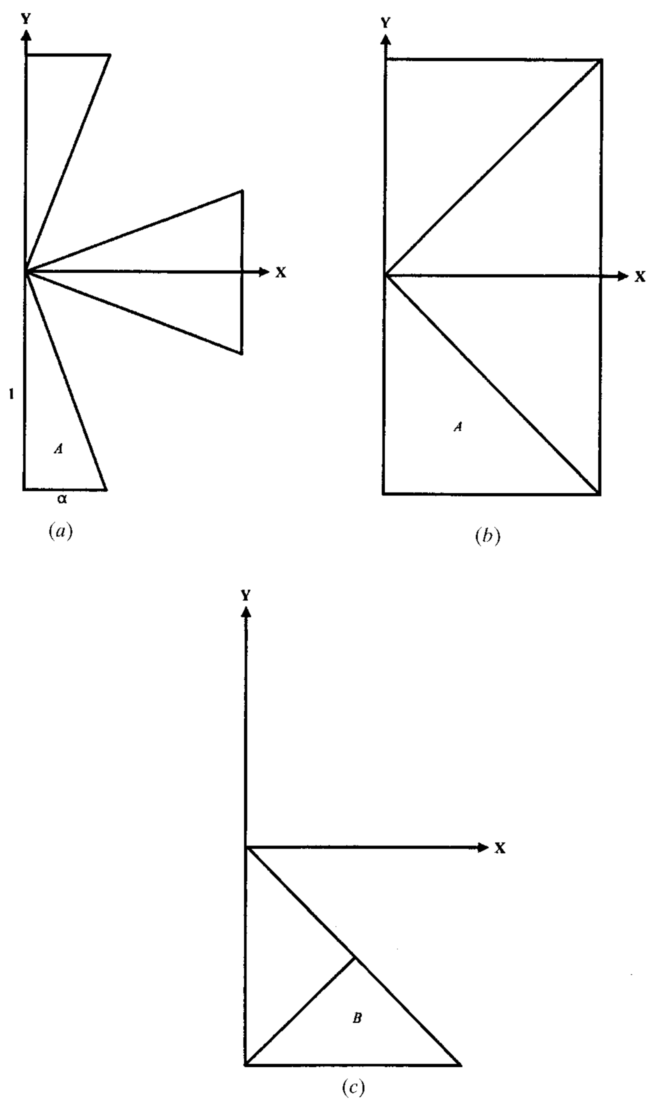

Figure 4
Composite formed from a basic component A and $\{g_1, g_2, g_3, g_4\} = \{1, m_y, m_{xy}, 4_z\}$ when (a) $\alpha < 1$ and (b) $\alpha = 1$. The subunit B of A is shown in (c).

$$\begin{aligned} S &= \{\ldots g_i h_j B \ldots\} \\ &= \{(1|0,0)B, (m_y|0,0)B, (m_{xy}|0,0)B, (4_z|0,0)B, \\ &\quad (m_{\bar{x}y}|1,-1)B, (4_z^3|1,1)B, (2_z|1,-1)B, (m_x|1,1)B\}. \end{aligned} \qquad (14)$$

The element $(m_x|1,0)$ of the point group $\mathbf{K}$ of B is in the normalizer of $\{\ldots g_i h_j \ldots\}$ modulo $\mathbf{K}$, and it follows that $\mathbf{K_s} = \{(1|0,0), (m_x|1,0)\}$ and the second invariance group, equation (7), is

$$\begin{aligned} \mathbf{Inv}(2) &= \langle \mathbf{Inv}(1), \mathbf{K_s} \rangle \\ &= \{(1|0,0), (m_y|0,0), (m_x|1,0), (2_z|1,0)\}. \end{aligned} \qquad (15)$$

This is the point group $\mathbf{2_z m_x m_y}(1/2,0)$, the point group $\mathbf{2_z m_x m_y}$ whose singular point is, in the coordinate system used in Fig. 4, at the coordinates $(1/2, 0)$.

The subunit B can be subdivided into symmetry-related subunits and these subunits further subdivided into smaller subunits. No new latent symmetries of the composite, however, are found among the symmetries of these subunits or smaller subunits, as none leave invariant the singular point at $(1/2, 0)$.

In this paper, we have shown that, in addition to common symmetries of the components of a composite and symmetries which permute the components, additional symmetries of the composite may be found among the symmetries of symmetry-related subunits of the components. We have also given a systematic method to determine such additional symmetries. It should not be construed, however, that one can determine *all* additional symmetries of the composite using this methodology. The reason for this is that the methodology presented searches for additional symmetry among the symmetries of *individual* symmetry-related subunits of a *single* component. Additional composite symmetries may be related to the symmetries of *combinations* of subunits of *different* components.

For example, Fig. 5 shows a composite constructed from a isosceles triangular prism denoted by A and a set of isometries $\{g_1, g_2, \ldots, g_8\}$ which constitute the group $\mathbf{4_z m_x m_{xy}}$. In Fig. 5(a), $\alpha < 1$ and we have a rectangular solid; in Fig. 5(b), $\alpha = 1$ and we have a cubic solid. The symmetry of the composite in Fig. 5(a) can be determined using the method given above, the cubic symmetry of the composite in Fig. 5(b) cannot. The reason for this is shown in Fig. 5(b). Additional cubic symmetry of the composite is not related to the symmetry of a symmetry-related subunit of A, but can be related to a combination of subunits of different components. To demonstrate this, in Fig. 5(b) we have shaded *four* subunits belonging to *two* components which show cubic symmetry.

3. Latent symmetry and the Curie–Shubnikov principle

According to the Curie principle of superposition of dissymmetries (see, for example, Shubnikov & Koptsik, 1974; Brandmuller, 1986; Wadhawan, 2000), when several phenomena of different origin are superimposed in one and the same system, their dissymmetries are summed. Let $\mathbf{G_1}$, $\mathbf{G_2}, \ldots$ denote the symmetry groups of the individual phenomena taken separately. The superimposed composite system is invariant under the group

$$\mathbf{G_d} = \mathbf{G_1} \cap \mathbf{G_2} \cap \ldots = \cap_i \mathbf{G_i}. \tag{16}$$

This is the process of *dissymmetrization* or *lowering of symmetry* (Shubnikov & Koptsik, 1974).

In certain situations, the symmetry group $\mathbf{G_s}$ of the superimposed composite system is higher than $\mathbf{G_d}$, *i.e.* $\mathbf{G_s}$ contains the group $\mathbf{G_d}$ as a subgroup. It is then said that a process of *symmetrization* or *symmetry enhancement* has occurred. The symmetry group $\mathbf{G_s}$ of the superimposed composite system is written in terms of its subgroup $\mathbf{G_d}$ as

$$\mathbf{G_s} = \mathbf{G_d} \cup g_2\mathbf{G_d} \cup \ldots \cup g_j\mathbf{G_d}. \tag{17}$$

These additional symmetries $g_2, g_3, \ldots, g_j$ of the composite are called *symmetrizers* (Shubnikov & Koptsik, 1974). The concept of latent symmetry can be used to determine such symmetrizers without prior knowledge of the symmetry group of the composite.

Consider the square composite in Fig. 2. The symmetry groups of the two components are identical, $\mathbf{G}_1 = \mathbf{G}_2 = \mathbf{m_x}$, and consequently $\mathbf{G_d} = \mathbf{m_x}$. The symmetry group of this composite is $\mathbf{G_s} = \mathbf{4_z m_x m_{xy}}$, $\mathbf{G_s}$ can be written as

$$\mathbf{G_s} = \mathbf{G_d} \cup 2_z\mathbf{G_d} \cup m_y\mathbf{G_d} \cup m_{xy}\mathbf{G_d}, \tag{18}$$

with symmetrizers 2_z, m_y and m_{xy}. Note that these symmetrizers were determined after the symmetry group of the composite was given.

As we have shown in the preceding section, the symmetry group of this composite, and consequently the above symmetrizers, can be determined in a systematic manner without any prior knowledge as to what is the symmetry group of the composite. It is with the latent symmetry of a subunit of the basic component of the composite with which one can predict such symmetrizers.

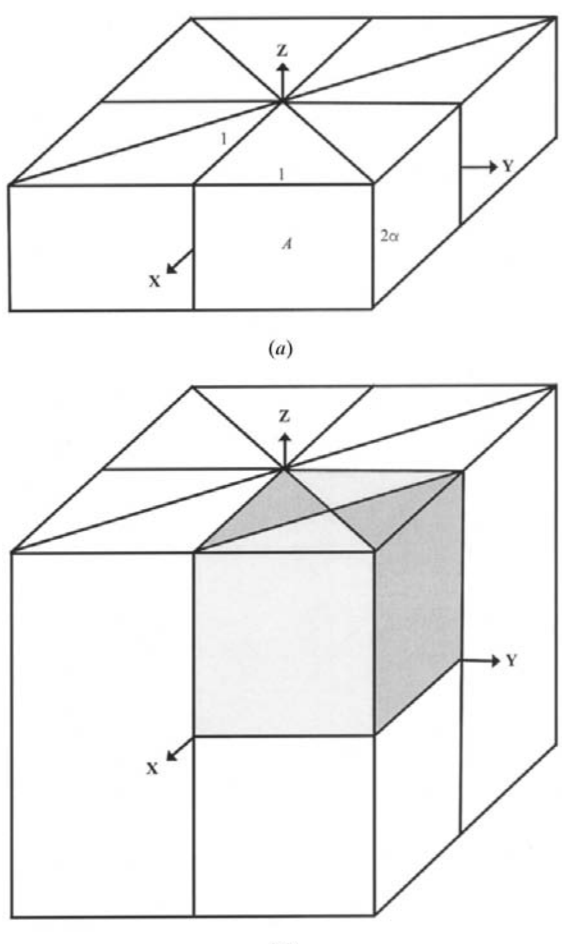

Figure 5
Composite formed from an isosceles triangular prism basic component A and isometries $\{g_1, g_2, \ldots, g_8\}$ which constitute the group $\mathbf{4_z}\boldsymbol{m_x m_{xy}}$ when (a) $\alpha < 1$ and (b) $\alpha = 1$. Shaded in (b) is a combination of four subunits of two different components which shows cubic symmetry.

4. Applications

This concept of latent symmetry is applicable in at least two important fields, namely in the field of the symmetry of composite systems and in the field of phase transitions. With respect to the former, there is much literature dealing with the symmetry of bicrystals with a homophase interface, relevant also to twinning, grain boundaries and domain structures [see Vlachavas (1984), and references therein, and Wadhawan (2000)]. Not taking into account the possibilities of latent symmetry can lead to erroneous conclusions. As pointed out by Wadhawan (1987), a theorem proved by Vlachavas (1984) is not correct. According to the theorem, the order of the composite symmetry group can at most be twice the order of the symmetry groups of the components. The square composite discussed above is a counterexample to that theorem: the order of the composite symmetry **4mm** is four times that of the order of the component symmetry **m**.

In the extended Landau theory of phase transitions, information about the crystal structure is introduced through the *tensor field criterion* (Birman, 1966; Litvin, 1982; Litvin *et al.*, 1982): the active irreducible representation responsible for a phase transition from the original (prototype) crystal structure of symmetry S_0 to the structure of the daughter phase must be contained in a tensor field representation of S_0. The latter, by definition, is a direct product of a tensor representation of S_0 and a permutation representation of the atoms of the crystal. The atomic structure changes with temperature and other control parameters. It is conceivable that for a certain set of control parameters the interatomic bond angles may acquire special values, leading to the manifestation of latent symmetry, and the concomitant 'symmetry jumps' in a sequence of phase transitions.

APPENDIX *A*

Consider a group **M** and an unordered subset of elements $\{g_1, g_2, \ldots, g_n\}$ of **M**. The set of all elements m of the group **M** which map by conjugation the unordered set $\{g_1, g_2, \ldots, g_n\}$ onto itself, *i.e.*

$$m\{g_1, g_2, \ldots, g_n\}m^{-1} = \{mg_1m^{-1}, mg_2m^{-1}, \ldots, mg_nm^{-1}\} = \{g_1, g_2, \ldots, g_n\}, \quad (19)$$

constitute a subgroup of **M** called the *normalizer of the subset* $\{g_1, g_2, \ldots, g_n\}$ in **M** (Kurosh, 1960).

Consider a subgroup $\mathbf{H} = \{h_1, h_2, \ldots\}$ of **M**. One can inquire as to what elements of the group **M** satisfy the following condition,

$$m\{g_1, g_2, \ldots, g_n\}m^{-1} = \{mg_1m^{-1}, mg_2m^{-1}, \ldots, mg_nm^{-1}\} = \{g_1h_1, g_2h_2, \ldots, g_nh_n\}. \quad (20)$$

That is, each element m maps the unordered subset $\{g_1, g_2, \ldots, g_n\}$ not necessarily onto itself, but onto a set where each element of the set $\{g_1, g_2, \ldots, g_n\}$ is multiplied on the right by a not necessarily distinct element of **H**. The subset of elements of **M** which satisfy (20) is at least as large as the normalizer of the subset $\{g_1, g_2, \ldots, g_n\}$ and in general does not constitute a group. The subset of all elements h of **H**, however, which satisfy (20), that is which satisfy the condition

$$h\{g_1, g_2, \ldots, g_n\}h^{-1} = \{hg_1h^{-1}, hg_2h^{-1}, \ldots, hg_nh^{-1}\} = \{g_1h_1, g_2h_2, \ldots, g_nh_n\}, \quad (21)$$

do constitute a group. We first show that the subset of all elements of **H** which satisfy (21) is closed and then that the inverse of each element in the subset is also contained in the subset. If h and h' satisfy (21), then it follows that

$$\begin{aligned} hh'\{g_1, g_2, \ldots, g_n\}h'^{-1}h^{-1} &= h\{h'g_1h'^{-1}, h'g_2h'^{-1}, \ldots, h'g_nh'^{-1}\}h^{-1} \\ &= h\{g_1h'_1, g_2h'_2, \ldots, g_nh'_n\}h^{-1} \\ &= \{hg_1h'_1h^{-1}, hg_2h'_2h^{-1}, \ldots, hg_nh'_nh^{-1}\} \\ &= \{hg_1h^{-1}hh'_1h^{-1}, hg_2h^{-1}hh'_2h^{-1}, \ldots, hg_nh^{-1}hh'_nh^{-1}\}. \end{aligned} \quad (22)$$

Writing $hg_ih^{-1} = g_jh_j$, then $hg_ih^{-1}hh'_ih^{-1} = g_jh_jhh'_ih^{-1} = g_jh''_j$ and

$$hh'\{g_1, g_2, \ldots, g_n\}h'^{-1}h^{-1} = \{g_1h''_1, g_2h''_2, \ldots, g_nh''_n\}. \quad (23)$$

Consequently, if h and h' satisfy (21) we have that the product hh' also satisfies this equation and the subset of all elements of **H** which satisfy (21) is closed under group multiplication.

If an element h of **H** satisfies (21), it permutes the set $\{g_1, g_2, \ldots, g_n\}$ by conjugation, *i.e.* g_i is permuted into g_j modulo an element of **H** *via*

$$hg_ih^{-1} = g_jh_j. \quad (24)$$

This can be rewritten as

$$g_j = hg_ih^{-1}h_j^{-1} \quad (25)$$

and

$$h^{-1}g_jh = g_ih^{-1}h_j^{-1}h. \quad (26)$$

Since h, h^{-1} and h_j^{-1} are all elements of **H**, we may write $h^{-1}h_j^{-1}h = h_i$ and

$$h^{-1}g_jh = g_ih_i. \quad (27)$$

That is, the inverse element h^{-1} permutes the set $\{g_1, g_2, \ldots, g_n\}$ by conjugation modulo an element of **H**, the inverse permutation of that engendered by the element h. Consequently, if h satisfies (21), it follows from (27) that h^{-1}, the inverse of h, also satisfies this equation,

$$h^{-1}\{g_1, g_2, \ldots, g_n\}h = \{h^{-1}g_1h, h^{-1}g_2h, \ldots, h^{-1}g_nh\} = \{g_1h_1, g_2h_2, \ldots, g_nh_n\}. \quad (28)$$

Since the subset of all elements of **H** which satisfies (21) is closed under group multiplication and the inverse of each element of the subset is also in the subset, the subset is a subgroup of **H**.

In this paper we are interested in such subgroups of a group **H** which satisfy (21) for a given subset of elements $\{g_1, g_2, \ldots, g_n\}$ of a group **M**. This is because of the theorem proven in Appendix *B* and its corollary given in Appendix *C* which are central in determining the latent symmetry of a subunit of a component of a composite. We shall say that all elements h of **H** which satisfy (21) are contained in the *normalizer of the set* $\{g_1, g_2, \ldots, g_n\}$ *in* **M** *modulo* **H**. In general we shall not explicitly refer to the group **M** and use the shorter phrase *in the normalizer of* $\{g_1, g_2, \ldots, g_n\}$ *modulo* **H**, and when the subset $\{g_1, g_2, \ldots, g_n\}$ constitutes a group **G**, use the phrase *in the normalizer of* **G** *modulo* **H**.

For example, consider the two groups $\mathbf{H} = \mathbf{2_x2_y2_z}$ and $\mathbf{G} = \mathbf{3_{xyz}}$ (here we can take **M** as the group of all rotations about a point). Calculating hgh^{-1} for all g and h (note that $h^{-1} = h$ for all h),

$$\begin{array}{lllllllll} 1\mathbf{G}1 & = & 1 & 3_{xyz} & 3^2_{xyz} & & & & \\ 2_x\mathbf{G}2_x & = & 1 & 3_{\bar{x}yz} & 3^2_{\bar{x}yz} & = & 1 & 3^2_{xyz}2_z & 3_{xyz}2_y \\ 2_y\mathbf{G}2_y & = & 1 & 3_{x\bar{y}z} & 3^2_{x\bar{y}z} & = & 1 & 3^2_{xyz}2_x & 3_{xyz}2_z \\ 2_z\mathbf{G}2_z & = & 1 & 3_{xy\bar{z}} & 3^2_{xy\bar{z}} & = & 1 & 3^2_{xyz}2_y & 3_{xyz}2_x. \end{array}$$

We have that only the unit element $h = 1$ of **H** is in the normalizer of **G**, *i.e.* satisfies (19). All elements of **H**, however, are in the normalizer of **G** modulo **H**, *i.e.* satisfy (21).

APPENDIX *B*

Given an object A of symmetry **H** and a set of isometries $\{g_1, g_2, \ldots, g_n\}$. We construct an unordered set S of objects by applying each of the isometries of $\{g_1, g_2, \ldots, g_n\}$ to the object A,

$$S = \{g_1A, g_2A, \ldots, g_nA\}. \quad (29)$$

We prove here the following theorem:

Theorem 1: The unordered set $S = \{g_1A, g_2A, \ldots, g_nA\}$ is invariant under an element h of **H**, the symmetry group of the object A, if and only if h is contained in the normalizer of $\{g_1, g_2, \ldots, g_n\}$ modulo **H**. All such elements of **H** constitute a subgroup $\mathbf{H_s}$ of **H**.

For the sufficiency of this theorem, we assume that an element h of **H** is contained in the normalizer of $\{g_1, g_2, \ldots, g_n\}$ modulo **H**, see Appendix *A*, and show that h leaves S invariant. Applying h to S we have

$$hS = h\{g_1A, g_2A, \ldots, g_nA\} = \{hg_1A, hg_2A, \ldots, hg_nA\}. \quad (30)$$

Since h is contained in the normalizer of $\{g_1, g_2, \ldots, g_n\}$ modulo **H**, for $i = 1, 2, \ldots, n, hg_ih^{-1} = g_jh_j$. Therefore, $hg_i = g_jh_jh = g_jh_j'$ and we can write

$$hS = \{g_1h_1'A, g_2h_2'A, \ldots, g_nh_n'A\} = \{g_1A, g_2A, \ldots, g_nA\}, \quad (31)$$

where we have used the fact that A is invariant under elements of **H**. Consequently, $hS = S$ and h leaves the set S invariant.

For the necessity of this theorem, we assume that an element h of **H** leaves the set S invariant and show that it must be contained in the normalizer of $\{g_1, g_2, \ldots, g_n\}$ modulo **H**. Writing the equation $hS = S$ explicitly, we have

$$\{hg_1A, hg_2A, \ldots, hg_nA\} = \{g_1A, g_2A, \ldots, g_nA\}. \quad (32)$$

The members g_iA of the set S are permuted by h. For $i = 1, 2, \ldots, n$, the ith member of S is permuted by h into the $j(i)$th member, that is

$$hg_iA = g_{j(i)}A \quad (33)$$

and

$$g_{j(i)}^{-1}hg_iA = A. \quad (34)$$

We introduce an identity operator $h^{-1}h$,

$$g_{j(i)}^{-1}hg_i(h^{-1}h)A = A. \quad (35)$$

Since $hA = A$, this becomes

$$g_{j(i)}^{-1}hg_ih^{-1}A = A, \quad (36)$$

and since the symmetry group of the object A is **H**, it follows that $g_{j(i)}^{-1}hg_ih^{-1}$ is an element h of **H**, an element which we denote by h_j,

$$g_{j(i)}^{-1}hg_ih^{-1} = h_j, \quad (37)$$

and consequently for $i = 1, 2, \ldots, n$,

$$hg_ih^{-1} = g_{j(i)}h_j. \quad (38)$$

Thus an element h of **H** which leaves the set S invariant belongs to the normalizer of $\{g_1, g_2, \ldots, g_n\}$ modulo **H**. That all elements of **H** which are in the normalizer of $\{g_1, g_2, \ldots, g_n\}$ modulo **H** constitute a subgroup $\mathbf{H_s}$ of **H** was proven in Appendix *A*. QED

If **H** is the symmetry group of A, then the symmetry group of the component g_iA of S is the group $g_i\mathbf{H}g_i^{-1}$. We note that all symmetry operations common to the symmetry groups of all the components are contained in the subgroup of elements of **H** in the normalizer of $\{g_1, g_2, \ldots, g_n\}$ modulo **H**. Let h denote an element common to the symmetry groups of all the components of S. We have

$$h\{g_1, g_2, \ldots, g_n\}h^{-1} = \{hg_1h^{-1}, hg_2h^{-1}, \ldots, hg_nh^{-1}\} \quad (39)$$

and since, for each $i = 1, 2, \ldots, n$, $h = g_ih_ig_i^{-1}$, we have $hg_ih^{-1} = g_ih_ig_i^{-1}g_ih^{-1} = g_ih_ih^{-1} = g_ih_i'$, it follows that

$$h\{g_1, g_2, \ldots, g_n\}h^{-1} = \{g_1h_1', g_2h_2', \ldots, g_nh_n'\} \quad (40)$$

and all symmetry operations common to the symmetry groups of *all* the components are contained in the subgroup of elements of **H** in the normalizer of $\{g_1, g_2, \ldots, g_n\}$ modulo **H**.

As an example, consider the case where $\mathbf{H} = \mathbf{4_z2_x2_{xy}}$ and the set $\{g_1, g_2, \ldots, g_n\}$ consisting of the elements of the group $\mathbf{G} = \mathbf{3_{xyz}}$. The symmetry groups of the components of the composite S are, respectively, $\mathbf{4_z2_x2_{xy}}$, $\mathbf{4_x2_y2_{yz}}$ and $\mathbf{4_y2_z2_{zx}}$. The symmetry common to these groups is $\mathbf{2_x2_y2_z}$. This group is contained in the group of all elements of **H** which are in the normalizer of **G** modulo **H**, which in this case is the group $\mathbf{H} = \mathbf{4_z2_x2_{xy}}$ itself.

APPENDIX *C*

If an object A has symmetry $\mathbf{H} = \{h_1, h_2, \ldots, h_m\}$, it follows that one can identify a subunit B in A such that $A = \{h_1B, h_2B, \ldots, h_mB\}$. Let **K** denote the symmetry group of B. The set $S = \{g_1A, g_2A, \ldots, g_nA\}$ defined in Appendix *B* can now, by substituting $A = \{h_1B, h_2B, \ldots, h_mB\}$, be written as

$$S = \{\ldots, g_ih_jB, \ldots\}, \quad (41)$$

where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. We have then that the set S can be considered as a set of objects obtained by applying all elements of the set $\{\ldots, g_ih_j, \ldots\}$, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, to the object B whose symmetry group is **K**. We have, as a corollary of the theorem in Appendix *B*, that *the unordered set* $S = \{\ldots, g_ih_jB, \ldots\}$, $i = 1, 2, \ldots, n$ *and* $j = 1, 2, \ldots, m$, *is invariant under an element k of* **K**, *the symmetry group of the object B, if and only if k is in the normalizer of the set* $\{\ldots, g_ih_j, \ldots\}$, $i = 1, 2, \ldots, n$ *and* $j = 1, 2, \ldots, m$, *modulo* **K**. *All such elements of* **K** *constitute a subgroup* $\mathbf{K_s}$ *of* **K**.

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